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Handles in Graphs and Sphere Bundles over S^1*

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We extend to $(n + 1)$-coloured graphs the concept of combinatorial handle, presented for $n = 3$, in [7], [11], [19]. Then we study the operation of cancelling of such a handle, which always reveals a connected sum decomposition of the represented manifolds.

1. INTRODUCTION AND BASIC NOTATIONS

Throughout this paper, the term *graph* will stand for *multigraph*: hence multiple edges are allowed, but loops are forbidden. We shall often set $\Gamma = (V, E)$ for a graph Γ, having $V = V(\Gamma)$ and $E = E(\Gamma)$ as sets of vertices and edges respectively; both sets are supposed to be finite. All graphs will be connected, unless otherwise stated. If Γ is a bipartite graph, then we shall set $\Gamma = (V', V''; E)$, in order to point out the two classes into which $V(\Gamma)$ splits.

By an $(n + 1)$-*coloured graph* we mean a pair (Γ, γ), where Γ is a graph, regular of degree $n + 1$, and γ: $E(\Gamma) \to \Delta_n = \{i \in \mathbb{Z} | 0 \leqslant i \leqslant n\}$ is an edge-colouration on Γ, by means of $n + 1$ colours (whence $\gamma(e) \neq \gamma(f)$, for all pairs e, f of adjacent edges of Γ). For each $\mathscr{B} \subseteq \Delta_n$, $\Gamma_{\mathscr{B}}$ will denote the subgraph $(V(\Gamma), \gamma^{-1}(\mathscr{B}))$; for any colour $c \in \Delta_n$, $\hat{c}$ will stand for $\Delta_n - \{c\}$.

For all notions from graph theory, which are not explicitly defined in this work, we refer to [9].

The basic notions and results from PL-topology, used in this paper, can be found in [16] and [6]. All considered spaces and maps will be in the polyhedral category. The prefix 'PL' will always be omitted. All manifolds will be connected, if not otherwise stated.

For the notion of *pseudocomplex* see [10, p. 49]. An n-dimensional pseudocomplex $K(\Gamma, \gamma)$—often simply indicated as $K(\Gamma)$—is uniquely associated to every $(n + 1)$-coloured graph (Γ, γ) (see [2], [6], [15]); the i-simplexes of $K(\Gamma)$ are in bijection with the connected components of the subgraphs $\Gamma_{\mathscr{B}}$, with Card $\mathscr{B} = n - i$. The graph (Γ, γ) is said to *represent* the space $|K(\Gamma)|$ and every homeomorphic polyhedron. If $\Gamma_{\hat{c}}$ is connected, for all $c \in \Delta_n$, then $K(\Gamma)$ has exactly $n + 1$ vertices (0-simplexes); in this case, both (Γ, γ) and $K(\Gamma)$ are said to be *contracted.*

A theorem of Pezzana [2], [14], [15] assures that every closed connected n-manifold M can be represented by at least one contracted $(n + 1)$-coloured graph (Γ, γ). Any one of such graphs (Γ, γ) is called a *crystallization* of M, and the associated $K(\Gamma)$ a *contracted triangulation* of M.

A general survey of the theory of edge-coloured graphs representing manifolds is contained in [5].

The concept of (*combinatorial*) *handle* in a 4-coloured graph (Γ, γ) was introduced in [7, defn 2]. It consists of two vertices P, Q, joined by two edges e_0, e_1, coloured by two colours 2 and 3 say, such that: (a) the cycle of $\Gamma_{\{0,1\}}$, which contains P, also contains Q, and (b) e_0 and e_1 are the only edges of Γ, which join P with Q. To alternative proofs of the following result are contained in [7, main theorem] and [11, main theorem]. *If (Γ, γ) is a crystallization of a closed 3-manifold M, containing a handle, then there exists a connected sum decomposition $M \simeq \tilde{M} \# \mathbb{H}$, where $\tilde{M}$ is a suitable 3-manifold, and $\mathbb{H}$ is either $\mathbb{S}^2 \times \mathbb{S}^1$ or $\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^1$ (the orientable or the non orientable 2-sphere bundle over $\mathbb{S}^1$).*

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A similar result for Heegaard splittings is contained in [13]. The neighbouring problem of characterizing the graphs which represent $\mathbb{S}^3$ is faced by means of a shelling algorithm in [18].

In this work, we extend the concept of handle, and the previous result to dimension n, showing that such a configuration (in a crystallization) always reveals a sphere bundle over $\mathbb{S}^1$, as a connected summand of the represented manifold. The proof is inspired by the one of [11, main thm] as modified by one of the authors in [19], where the first step of the inductive argument presented here had already been developed.

2. Sphere Bundles over $\mathbb{S}^1$

As it is well known (see e.g. [17, sect. 26]), for every $n \geqslant 2$ there are exactly two $\mathbb{S}^{n-1}$-bundles over $\mathbb{S}^1$, i.e. the trivial one $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and the 'twisted' one $\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1$. The latter (resp. the former) can be thought of as obtained from $\mathbb{S}^{n-1} \times [0, 1]$, by identifying $\mathbb{S}^{n-1} \times \{0\}$ with $\mathbb{S}^{n-1} \times \{1\}$, under an orientation preserving (resp. reversing) homeomorphism.

The following Definition 1 and Lemma 2 are the obvious extensions of the analogous concepts introduced in [12] and [11].*

Definition 1. Two vertices A', A'' of an $(n + 1)$-coloured graph (Γ, γ) are said to be *completely separated* iff, for every colour $c \in \Delta_n$, A' and A'' belong to different components of $\Gamma_{\hat{c}}$.

Given an $(n + 1)$-coloured graph (Γ, γ), and two vertices A', $A'' \in V(\Gamma)$, the graph $(\Gamma\text{fus}(A', A''), \gamma\,\text{fus}(A', A''))$ is defined exactly as for $n = 3$ (see [12]): (a) $V(\Gamma\text{fus}(A', A'')) = V(\Gamma) - \{A', A''\}$; (b) two vertices X, Y are joined in $\Gamma\text{fus}(A', A'')$ by an edge coloured c, iff either X, Y were joined in Γ by such an edge, or X (resp. Y) was joined in Γ to A' (resp. A''), by an edge coloured c.

Remark 1. If (Γ, γ) has two components, (Γ', γ') and (Γ'', γ'') say, and $A' \in V(\Gamma')$, $A'' \in V(\Gamma'')$, then the described graph $\Gamma\text{fus}(A', A'')$ coincides with the 'connected sum' $\Gamma' \#_{A'A''} \Gamma''$ of Γ' and Γ'', with respect to A' and A'' (see [5, sect. 3]); hence, if (Γ', γ') and (Γ'', γ'') represent two closed n-manifolds M', M'', then $(\Gamma\text{fus}(A', A''), \gamma\,\text{fus}(A', A''))$ represents $M' \# M''$. Observe that, in this case, A' and A'' are obviously completely separated.

Lemma 2. *If (Γ, γ) represents a closed (connected) n-dimensional manifold M, and A', A'' are two completely separates vertices of Γ, then $(\Gamma\text{fus}(A', A''), \gamma\,\text{fus}(A', A''))$ represents either $M \# (\mathbb{S}^{n-1} \times \mathbb{S}^1)$ or $M \# (\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1)$.*

Proof. Let $\sigma(A')$ and $\sigma(A'')$ be the n-simplexes of $K(\Gamma)$ corresponding to the vertices A' and A'' of Γ. A' and A'' being completely separated, $\sigma(A')$ and $\sigma(A'')$ have no 0-faces —thus also no h-faces—in common. The pseudocomplex $\check{K}$ obtained by deleting $\sigma(A')$ and $\sigma(A'')$ from $K(\Gamma)$, triangulates the manifold $\check{M}$, obtained by digging the interiors of two closed n-balls out of M. Hence $\Gamma\text{fus}(A', A'')$ represents a closed n-manifold $M^\$$, obtained by identifying together the two spherical components of $\partial\check{M}$, under a homeomorphism, i.e. $M^\$ \simeq M \# \mathbb{S}^{n-1} \times \mathbb{S}^1$ or $M^\$ \simeq M \# \mathbb{S}^{-1} \underset{\sim}{\times} \mathbb{S}^1$.

Remark 2. If both Γ and $\Gamma\text{fus}(A', A'')$ are bipartite graphs, then both M and $M^\$$ are orientable manifolds [proposition 15], [5, sect. 3], and $M^\$ \simeq M \# (\mathbb{S}^{n-1} \times \mathbb{S}^1)$.

*In [11] and [12] the following terminology is adopted: A *3-gem* is a 4-coloured graph representing a closed 3-manifold; a *simple 3-gem* is a 3-manifold crystallization.

If Γ is bipartite and $\Gamma \mathrm{fus}(A', A'')$ non-bipartite, then M is orientable, $M^\$ \simeq M \# (\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1)$.

Finally, if Γ is non-bipartite, then $\Gamma \mathrm{fus}(A', A'')$ must be non-bipartite; M and $M^\$$ are therefore non orientable, and $M^\$ \simeq M \# (\mathbb{S}^{n-1} \times \mathbb{S}^1) \simeq M \# (\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1)$.

COROLLARY 3. *Let* (Γ, γ), A' *and* A'' *be as in Lemma 2. If* $\Gamma = (V', V''; E)$ *is a bipartite graph, and* $A' \in V'$, $A'' \in V''$ *(resp.* $A', A'' \in V'$*), then* $(\Gamma \,\mathrm{fus}\,(A', A''), \gamma \,\mathrm{fus}\,(A', A''))$ *represents* $M \# (\mathbb{S}^{n-1} \times \mathbb{S}^1)$ *(resp.* $M \# (\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1)$*).*

PROOF. If Γ is bipartite, and $A' \in V'$, $A'' \in V''$, then $\Gamma \mathrm{fus}(A', A'')$ is again bipartite, while if $A', A'' \in V'$, then $\Gamma \mathrm{fus}(A', A'')$ is non-bipartite, as it is easy to check. The statement is now a simple consequence of the previous Remark 2.

As an application, we now describe a general procedure to get a crystallization of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and of $\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1$, for all $n \geqslant 2$.

Let $(\Omega^{(n+1)}, \omega^{(n+1)})$ be the $(n + 1)$-coloured graph, representing $\mathbb{S}^n$, obtained by induction as follows:
(a) Call $(\Omega^{(0)}, \omega^{(0)})$ the standard crystallization of $\mathbb{S}^n$, formed by two vertices P_0, Q_0, joined by $n + 1$ (differently coloured) edges; (b) for $1 \leqslant h \leqslant n + 1$, define $(\Omega^{(h)}, \omega^{(h)})$ to be the graph obtained by adding a dipole of type n [3, sect. 3], with vertices P_h, Q_h, on the only edge coloured $h - 1$ of $\Omega^{(h-1)}$, with ends P_{h-1}, Q_{h-1}.

As it is easy to see, all the graphs $(\Omega^{(h)}, \omega^{(h)})$ represent $\mathbb{S}^n$, and are obviously bipartite, with $2(h + 1)$ vertices P_j, Q_j, $j \in \Delta_h$. If we set $\Omega^{(h)} = (V'^{(h)}, V''^{(h)}; E^{(h)})$, let us suppose that $V'^{(h)} = \{P_j \mid j \in \Delta_h\}$, $V''^{(h)} = \{Q_j \mid j \in \Delta_h\}$.

COROLLARY 4. *With the above notations, the graphs* $(\Omega^{(n+1)} \,\mathrm{fus}(P_0, Q_{n+1}), \omega^{(n+1)} \,\mathrm{fus}(P_0, Q_{n+1}))$ *and* $(\Omega^{(n+1)} \,\mathrm{fus}(P_0, P_{n+1}), \omega^{(n+1)} \,\mathrm{fus}(P_0, P_{n+1}))$ *are two crystallizations of* $\mathbb{S}^{n-1} \times \mathbb{S}^1$ *and* $\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1$ *respectively.*

PROOF. Both graphs are contracted; in order to show that they represent the two $\mathbb{S}^{n-1}$-bundles over $\mathbb{S}^1$, let us observe that in $\Omega^{(n+1)}$, P_0 is completely separated from both P_{n+1} and Q_{n+1}. The result follows now from Corollary 3.

The previously described crystallizations coincide with the ones presented in [4, sect. 2], for the orientable case. Figure 1 shows the whole procedure, for $n = 4$.

3. MAIN RESULTS

Let (Γ, γ) be an $(n + 1)$-coloured graph, and θ a subgraph of Γ, formed by two vertices P, Q, joined by $n - 1$ edges $e_1, \ldots, e_{n-1}$; if $\gamma(\theta)$ denotes the set $\{c \in \Delta_n \mid \exists e \in E(\theta), \gamma(e) = c\}$, and $\{i, j\} = \Delta_n - \gamma(\theta)$, then we shall call $C_{ij}(P)$ [resp. $C_{ij}(Q)$] the component of $\Gamma_{\{i,j\}}$—i.e. the cycle of Γ, with edges alternatively coloured i and j—which contains P (resp. Q).

DEFINITION 5. With the above notations, θ will be called a *(combinatorial) handle* in (Γ, γ) iff:
(a) $C_{ij}(P) = C_{ij}(Q)$* and
(b) $e_1, \ldots, e_{n-1}$ are the only edges which have P and Q as ends.

The previous definition extends the analogous ones of [7, defn 2], for $n = 3$, and of [Defn 2], for $n = 2$.

*Recall that if $C_{ij}(P) \neq C_{ij}(Q)$, then θ is called a dipole of type $n - 1$ in [3, sect. 3].

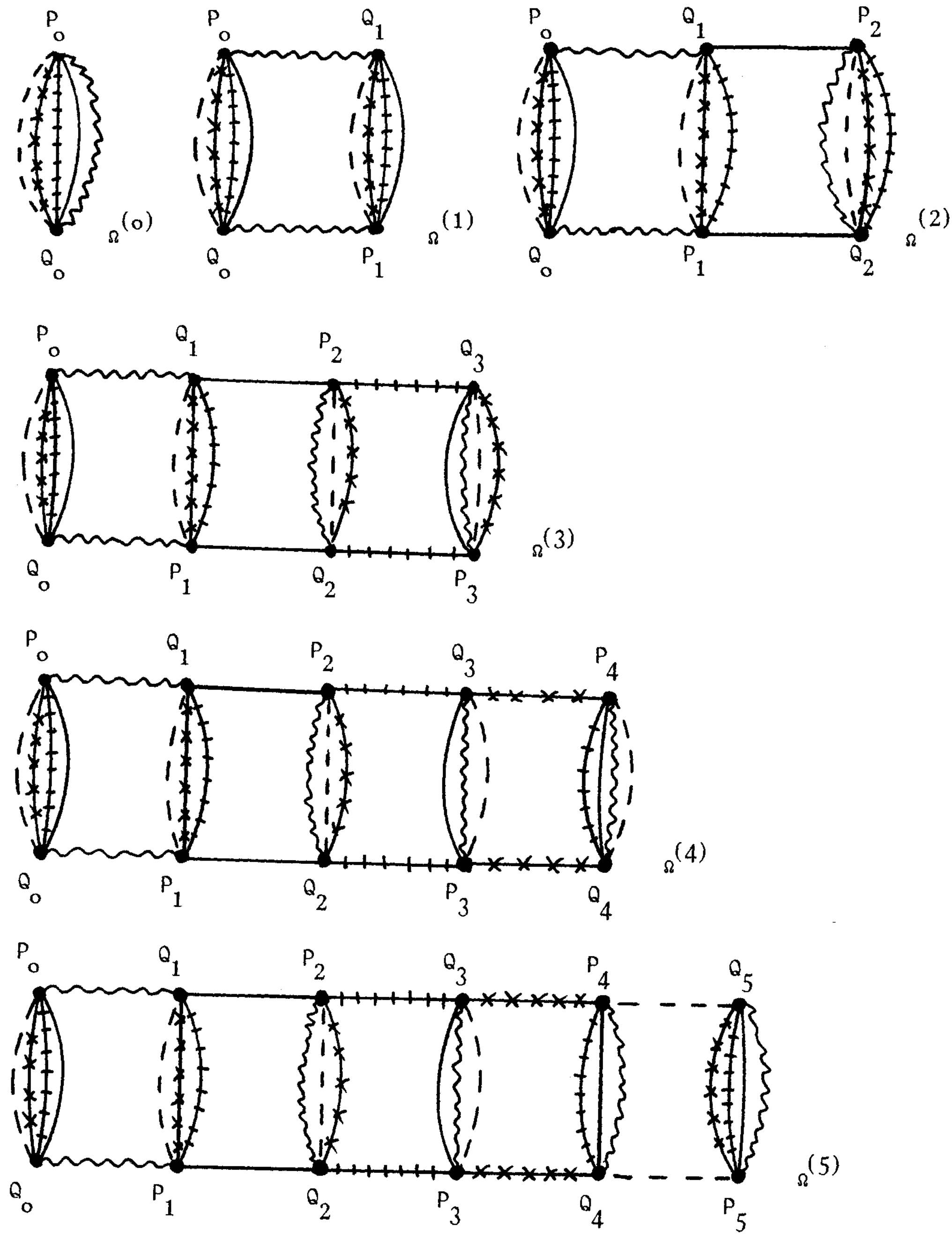

FIGURE 1.

Observe that the two crystallizations of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and $\mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1$, described in the preceding section, contain $n + 1$ such handles.

Let now θ be a handle in (Γ, γ), with vertices P, Q, and colour-set $\gamma(\theta) = \Delta_n - \{i, j\}$; let us call $C(\theta)$ $(= C_{ij}(P) = C_{ij}(Q))$ the cycle of $\Gamma_{\{i,j\}}$, containing P and Q. The two vertices split $C(\theta)$ into two paths $\mathscr{T}^{(1)}(\theta)$ and $\mathscr{T}^{(2)}(\theta)$, both with an even or an odd number of edges, since $C(\theta)$ is an even cycle. The handle θ will be called *proper* (resp. *improper*) iff $\mathscr{T}^{(1)}(\theta)$ and $\mathscr{T}^{(2)}(\theta)$ have an odd (resp. an even) number of edges.

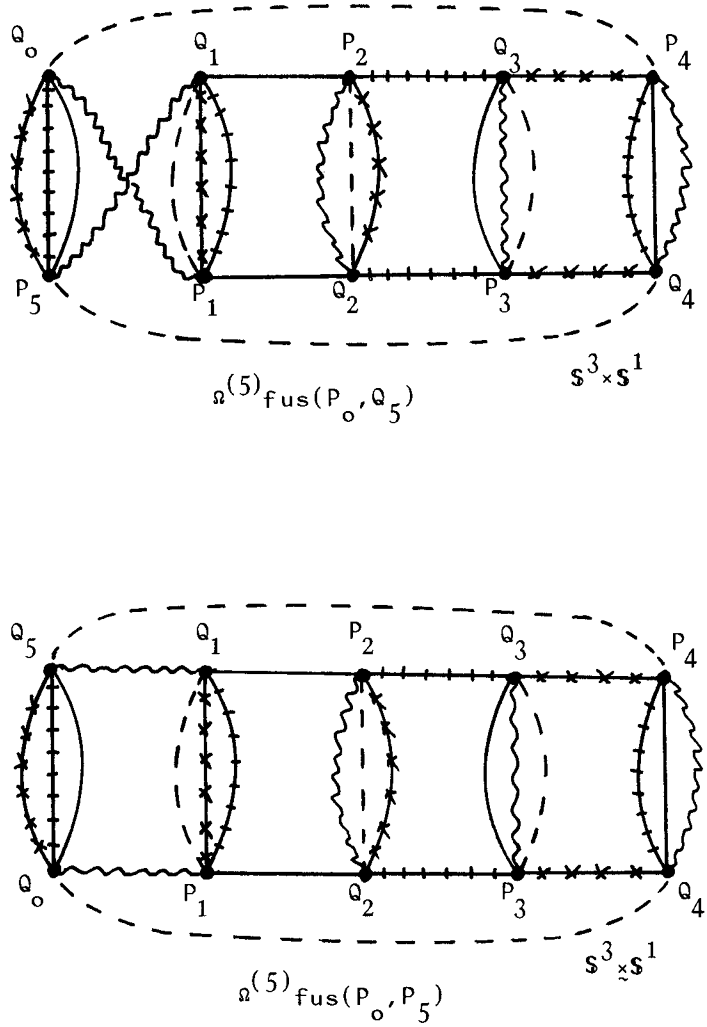

FIGURE 1. (continued).

Examples of improper handles, for $n = 2$, can be found in [figs 2b and 4a″].

As it is easy to see, a handle in a *bipartite* graph (Γ, γ) must be proper, since Γ cannot have odd cycles. As a consequence, we can state the following:

LEMMA 6. *Let θ be a handle in an $(n + 1)$-coloured graph (Γ, γ), with $n > 2$. If (Γ, γ) represents a closed n-manifold, then θ is proper.*

PROOF. Let c be a colour of $\gamma(\theta)$, and e be the only edge of θ coloured c. Each component Ξ of the n-coloured graph $(\Gamma_{\hat{c}}, \gamma_{\hat{c}})$, where $\gamma_{\hat{c}} = \gamma | E(\Gamma_{\hat{c}})$, represents S^{n-1} (since Γ represents a closed manifold of dimension n); $\Gamma_{\hat{c}}$ is therefore a bipartite graph [5, sect. 3]. In particular, if Ξ is the component of $\Gamma_{\hat{c}}$, which contains the vertices P and Q, then (if $n > 2$) Ξ contains a handle $\theta_{\hat{c}} = \theta - \{e\}$, with $\gamma_{\hat{c}}(\theta_{\hat{c}}) = \gamma(\theta) - \{c\}$, $C(\theta_{\hat{c}}) = C(\theta)$, $\mathcal{T}^{(h)}(\theta_{\hat{c}}) = \mathcal{T}^{(h)}(\theta)$, for $h = 1, 2$; it follows that θ and $\theta_{\hat{c}}$ are both proper or both improper handles. The statement of Lemma 6 is now a direct consequence of being $\theta_{\hat{c}}$ a handle in a bipartite graph.

We shall now describe a general procedure to construct a new $(n+1)$-coloured graph $(\tilde{\Gamma}, \tilde{\gamma})$ from a (Γ, γ), containing a handle θ. If P_i, Q_i (resp. P_j, Q_j) are the vertices of Γ, joined to P and Q by an edge coloured i (resp. j), then $(\tilde{\Gamma}, \tilde{\gamma})$ is obtained simply by deleting P and Q from Γ, together with all edges adjacent with them, and by joining P_i and Q_i (resp. P_j and Q_j) with an edge e_i (resp. e_j), coloured i (resp. j).

The graph $(\tilde{\Gamma}, \tilde{\gamma})$ will be said to be obtained from (Γ, γ) by *cancelling the handle* θ.

Observe that the graph $\tilde{\Gamma}$ may be disconnected, with at most two connected components.

For $n = 2, 3$, the previous operation has been introduced in [19], where its geometrical meaning was explored [19, propositions 2 and 3]. The n-dimensional analogue of the above quoted result constitutes the main purpose of the present paper.

THEOREM 7. *Let M_n be a closed (connected) n-dimensional manifold ($n \geqslant 3$), $(\Gamma^{(n)}, \gamma^{(n)})$ an $(n+1)$-coloured graph representing M_n, $\theta^{(n)}$ a handle in $\Gamma^{(n)}$, and $(\tilde{\Gamma}^{(n)}, \tilde{\gamma}^{(n)})$ the graph obtained by cancelling $\theta^{(n)}$ from $(\Gamma^{(n)}, \gamma^{(n)})$.*

(a) *If $(\tilde{\Gamma}^{(n)}, \tilde{\gamma}^{(n)})$ is connected, then it represents a closed (connected) n-manifold $\tilde{M}_n$, such that $M_n \simeq \tilde{M}_n \# \mathbb{H}_n$, where either $\mathbb{H}_n = \mathbb{S}^{n-1} \times \mathbb{S}^1$ or $\mathbb{H}_n = \mathbb{S}^{n-1} \underset{\sim}{\times} \mathbb{S}^1$.*

(b) *If $(\tilde{\Gamma}^{(n)}, \tilde{\gamma}^{(n)})$ has two connected components, $(\tilde{\Gamma}'^{(n)}, \tilde{\gamma}'^{(n)})$, $(\tilde{\Gamma}''^{(n)}, \tilde{\gamma}''^{(n)})$ say, then they represent two closed (connected) n-manifolds, $\tilde{M}_n'$, $\tilde{M}_n''$ say, such that $M_n \simeq \tilde{M}_n' \# \tilde{M}_n''$.**

The proof will be given by induction on n, and will need the following two sequences of Lemmas.

LEMMA 8_n. *Let $(\Sigma^{(n)}, \sigma^{(n)})$ be an $(n+1)$-coloured graph, representing $\mathbb{S}^n$ ($n \geqslant 2$), $\theta^{(n)}$ a handle in $\Sigma^{(n)}$, and $(\tilde{\Sigma}^{(n)}, \tilde{\sigma}^{(n)})$ the graph obtained by cancelling $\theta^{(n)}$ from $(\Sigma^{(n)}, \sigma^{(n)})$. Then $(\tilde{\Sigma}^{(n)}, \tilde{\sigma}^{(n)})$ has exactly two connected components, $(\tilde{\Sigma}'^{(n)}, \tilde{\sigma}'^{(n)})$, $(\tilde{\Sigma}''^{(n)}, \tilde{\sigma}''^{(n)})$ say, both representing $\mathbb{S}^n$.*

Let now $(\Gamma^{(n)}, \gamma^{(n)})$ be an $(n+1)$-coloured graph, $\theta^{(n)}$ a handle in $\Gamma^{(n)}$, and $(\tilde{\Gamma}^{(n)}, \tilde{\gamma}^{(n)})$ the graph obtained by cancelling $\theta^{(n)}$ from $(\Gamma^{(n)}, \gamma^{(n)})$. Let also P, Q be the vertices of $\theta^{(n)}$, P_i, Q_i (resp. P_j, Q_j) the vertices of $\Gamma^{(n)}$ joined to P and Q by an edge coloured i (resp. j), where $\{i, j\} = \Delta_n - \gamma^{(n)}(\theta^{(n)})$; finally, let $\tilde{e}_i$, $\tilde{e}_j$ be the two edges of $\tilde{\Gamma}^{(n)}$, with ends P_i, Q_i and P_j, Q_j, respectively, which proceed from the cancelling of the handle $\theta^{(n)}$.

Let us call $(\bar{\Gamma}^{(n)}, \bar{\gamma}^{(n)})$ the (possibly disconnected) graph, obtained from $(\tilde{\Gamma}^{(n)}, \tilde{\gamma}^{(n)})$ by adding two dipoles of type n, $\theta'^{(n)}$ and $\theta''^{(n)}$ say, on $\tilde{e}_i$ and $\tilde{e}_j$, respectively; let also A' (resp. A'') be the vertex of $\theta'^{(n)}$ (resp. $\theta''^{(n)}$), adjacent with P_i (resp. Q_j). Observe that the construction of $\bar{\Gamma}^{(n)}$, for $n = 3$, has been introduced in [11, proof of the main thm].

LEMMA 9_n. *Let M_n, $(\Gamma^{(n)}, \gamma^{(n)})$, $\theta^{(n)}$, $(\tilde{\Gamma}^{(n)}, \tilde{\gamma}^{(n)})$ be as in the hypothesis of Theorem 7, and $(\bar{\Gamma}^{(n)}, \bar{\gamma}^{(n)})$, A', A'' be the graph and the vertices described above. Then:*

(*a*) *$(\bar{\Gamma}^{(n)}, \bar{\gamma}^{(n)})$ represents a (possibly disconnected) closed n-manifold $\bar{M}_n$;*

(*b*) *A' and A'' are completely separated vertices of $(\bar{\Gamma}^{(n)}, \bar{\gamma}^{(n)})$, and ($\bar{\Gamma}^{(n)}$ fus (A', A''), $\bar{\gamma}^{(n)}$ fus (A', A'')) coincides with $(\Gamma^{(n)}, \gamma^{(n)})$.*

PROOF OF LEMMAS 8_n AND 9_n. Lemma 8_n is nothing but corollary 1 of [19]. We shall now prove that, for each $n \geqslant 3$, Lemma 9_n is a consequence of Lemma 8_{n-1}, and that Lemma 8_n comes from Lemma 9_n.

Suppose we have now proved Lemma 8_{n-1}. Let M_n, $(\Gamma^{(n)}, \gamma^{(n)})$, $\theta^{(n)}$, $(\tilde{\Gamma}^{(n)}, \tilde{\gamma}^{(n)})$, $(\bar{\Gamma}^{(n)}, \bar{\gamma}^{(n)})$, A' and A'' be as in Lemma 9_n. (a) In order to prove that $(\bar{\Gamma}^{(n)}, \bar{\gamma}^{(n)})$ represents a closed n-manifold $\bar{M}_n$, we shall show that, for each colour $c \in \Delta_n$, all components of $\bar{\Gamma}_{\hat{c}}^{(n)}$ represent $\mathbb{S}^{n-1}$.

Since $\Gamma^{(n)}$ represents M_n by hypothesis, each component of $\Gamma_{\hat{c}}^{(n)}$ ($c \in \Delta_n$) represents S^{n-1}. Now, if $c \notin \gamma^{(n)}(\theta^{(n)})$, then $\bar{\Gamma}_{\hat{c}}^{(n)}$ has exactly one more component than $\Gamma_{\hat{c}}^{(n)}$. Each component

*This topological meaning justifies the term 'handle', even when its cancellation increases the number of components of the graph.

of $\Gamma_{\hat{c}}^{(n)}$ is isomorphic to a component of $\bar{\Gamma}_{\hat{c}}^{(n)}$; the last component of $\bar{\Gamma}_{\hat{c}}^{(n)}$ has two vertices, joined by n (differently coloured) edges. Hence all components of $\bar{\Gamma}_{\hat{c}}^{(n)}$ represent S^{n-1}.

On the other hand, if $c \in \gamma^{(n)}(\theta^{(n)})$, then there is a component Ξ of $\Gamma_{\hat{c}}^{(n)}$, which contains the handle $\theta_{\hat{c}}^{(n)}$, obtained by deleting from $\theta^{(n)}$ the only edge coloured c; each component of $\Gamma_{\hat{c}}^{(n)}$, but Ξ, is isomorphic to a component of $\bar{\Gamma}_{\hat{c}}^{(n)}$. The graph Ξ, with the induced coloration, is an n-coloured graph representing $\mathbb{S}^{n-1}$: the cancelling of $\theta_{\hat{c}}^{(n)}$ from Ξ thus produces two components Ξ', Ξ'', both representing $\mathbb{S}^{n-1}$, by Lemma 8_{n-1}. By the last two components of $\bar{\Gamma}_{\hat{c}}^{(n)}$ are obtained from Ξ' and Ξ', by adding two dipoles of type $n - 1$. It follows that all components of $\bar{\Gamma}_{\hat{c}}^{(n)}$, c being any colour of Δ_n, represent $\mathbb{S}^{n-1}$. This concludes the proof of the point (a) of Lemma 9_n.

We have now to prove that A' and A'' are completely separated vertices of $\Gamma^{(n)}$. It is easy to see that they belong to different components of $\Gamma_{\hat{i}}^{(n)}$ and $\Gamma_{\hat{j}}^{(n)}$. Let now consider the subgraph $\Gamma_{\hat{c}}^{(n)}$, for $c \neq i, j$, and the component Ξ, containing the handle $\theta_{\hat{c}}^{(n)}$. As above, Ξ represents S^{n-1}, and the cancelling of $\theta_{\hat{c}}^{(n)}$ from it produces two components Ξ', Ξ'', by Lemma 8_{n-1}. By construction, A' and A'' are vertices of dipoles added on two edges belonging to Ξ' and Ξ'', respectively.

Besides, it is easy to check by direct construction that $(\Gamma^{(n)}, \gamma^{(n)})$ coincides with $[\bar{\Gamma}^{(n)} \text{ fus } (A', A''), \bar{\gamma}^{(n)} \text{ fus } (A', A'')]$, whence also the proof of the point (b) is completed.

Assume now Lemma 9_n. Let $(\Sigma^{(n)}, \sigma^{(n)})$ be an $(n + 1)$-coloured graph, representing $\mathbb{S}^n$, and $\theta^{(n)}$ a handle in $\Sigma^{(n)}$. Let also $(\tilde{\Sigma}^{(n)}, \tilde{\sigma}^{(n)})$ be the graph by cancelling $\theta^{(n)}$ from $\Sigma^{(n)}$ $(\bar{\Sigma}^{(n)}, \bar{\sigma}^{(n)})$ and A', A'' be the graph and the vertices obtained from $\tilde{\Sigma}^{(n)}$, by the construction described before stating Lemma 9_n. By Lemma 9_n to (a), $(\bar{\Sigma}^{(n)}, \bar{\sigma}^{(n)})$ represents a closed n-manifold $\bar{M}_n$; moreover, Lemma 9_n (b) assures that A', A'' are completely separated, and that $(\Sigma^{(n)}, \sigma^{(n)}) = (\bar{\Sigma}^{(n)} \text{ fus } (A', A''), \bar{\sigma}^{(n)} \text{ fus } (A', A''))$.

If $\bar{\Sigma}^{(n)}$ is connected, then the manifold represented by $\Sigma^{(n)}$ must have a $\mathbb{S}^{n-1}$-bundle over $\mathbb{S}^1$ as a connected summand, by Lemma 2; but this is absurd, as $\Sigma^{(n)}$ represents $\mathbb{S}^n$ by hypothesis. Hence $(\bar{\Sigma}^{(n)}, \bar{\sigma}^{(n)})$ must have two connected components $(\bar{\Sigma}'^{(n)}, \bar{\sigma}'^{(n)})$, $(\bar{\Sigma}''^{(n)}, \bar{\sigma}''^{(n)})$ say, representing two closed n-manifolds $\bar{M}_n'$, $\bar{M}_n''$, with $A' \in V(\bar{\Sigma}'^{(n)})$, $A'' \in V(\bar{\Sigma}''^{(n)})$. Now, since $\mathbb{S}^n$ must be homeomorphic with $\bar{M}_n' \# \bar{M}_n''$, both $\bar{M}_n'$ and $\bar{M}_n''$ are homeomorphic with $\mathbb{S}^n$ itself.

The statement of Lemma 8_n follows now easily from the above arguments, recalling that $\tilde{\Sigma}^{(n)}$ and $\bar{\Sigma}^{(n)}$ represent the same polyhedron, as the latter is obtained from by adding two dipoles of type n to the former.

This concludes the proof.

The proof of Theorem 7 is now a direct consequence of Lemma 9_n, and of Remark 1 and Lemma 2.

As it is easy to see, if (Γ, γ) is a contracted graph, then the cancelling of a handle from Γ cannot split the graph; hence, if we assume this additional hypothesis in Theorem 7, then only the case (a) may occur. This simple fact implies the following Corollary, which extends to dimension n the main result of [7] and [11].

COROLLARY 10. *Let (Γ, γ) be a crystallization of a closed n-manifold M. If (Γ, γ) admits a handle, then $M \simeq \tilde{M} \# \mathbb{H}$, where $\mathbb{H}$ is a $\mathbb{S}^{n-1}$-bundle over $\mathbb{S}^1$, and $\tilde{M}$ is a suitable manifold.*

Observe that the statement of Theorem 7 (a) also gives further informations on the manifold $\tilde{M}$.

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